$$2\begin{pmatrix}2\\3\end{pmatrix} + 3\begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}4\\5\end{pmatrix} = \begin{pmatrix}11\\14\end{pmatrix} \iff \begin{bmatrix}2 & 1 & 4\\3 & 1 & 5\end{bmatrix}\begin{bmatrix}2\\3\\1\end{bmatrix} = \begin{pmatrix}11\\14\end{pmatrix}$$

$$2\vec{a}_1 + 3\vec{a}_2 + \vec{a}_3 = \vec{b} \iff [\vec{a}_1\ \vec{a}_2\ \vec{a}_3]\begin{pmatrix}2\\3\\1\end{pmatrix} = \vec{b}$$

But $\vec{a}_1 + 2\vec{a}_2 - \vec{a}_3 = 0$

Case 1 put $x_1 = 0$. $\vec{a}_1 = \vec{a}_3 - 2\vec{a}_2$

$$\therefore\ 2(\vec{a}_3 - 2\vec{a}_2) + 3\vec{a}_2 + \vec{a}_3 = \vec{b}$$

$$-\vec{a}_2 + 3\vec{a}_3 = \vec{b}$$

$$[\vec{a}_1\ \vec{a}_2\ \vec{a}_3]\begin{bmatrix}0\\-1\\3\end{bmatrix} = \vec{b} \iff A\vec{x} = \vec{b}$$

Thus $\vec{x} = \begin{pmatrix}0\\-1\\3\end{pmatrix}$ is a basic solution corresponding to $\vec{a}_2$ & $\vec{a}_3$

But it is not feasible as $x_2 = -1 < 0$.

Case 2 : put $x_3 = 0$ $\vec{a}_3 = \vec{a}_1 + 2\vec{a}_2$

$$\therefore\ 2\vec{a}_1 + 3\vec{a}_2 + \vec{a}_1 + 2\vec{a}_2 = \vec{b}$$

$$\Rightarrow 3\vec{a}_1 + 5\vec{a}_2 = \vec{b}$$

Thus $\vec{x} = \begin{pmatrix}3\\5\\0\end{pmatrix}$ is a basic solution corresponding to $\vec{a}_1$ & $\vec{a}_2$

It is also feasible

$$A\vec{x} = \vec{b} \iff {}_m\big[\underbrace{B}_{m}\ \overset{n}{|}\ R\big]\begin{bmatrix}\vec{x}_B\\ \vec{x}_R\end{bmatrix} = \vec{b}$$

$\boxed{\text{rank}(A) = m.}$

$$\iff B\vec{x}_B + R\vec{x}_R = \vec{b}$$

Basic $(\vec{x}_R = \vec{0})$ (B invertible) $\Rightarrow B\vec{x}_B = \vec{b}$

$\Rightarrow \vec{x}_B = B^{-1}\vec{b} \Rightarrow$ basic solution $= \begin{bmatrix}B^{-1}\vec{b}\\0\end{bmatrix}$ feasible if $B^{-1}\vec{b} \geq \vec{0}$

Thm 2.2 FS $\hookrightarrow$ BFS.

Pf: W.l.o.g.

$$[\vec{a}_1, \vec{a}_2 \cdots, \vec{a}_p, \vec{a}_{p+1}, \ldots, \vec{a}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_p \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{b} \qquad (1)$$

$$x_i \geq 0 \qquad (2)$$

Case 1 $\{\vec{a}_1, \ldots \vec{a}_p\}$'s linearly independent.

(i) $p = m$:

$$(1) \Leftrightarrow [B \mid R] \begin{pmatrix} x_1 \\ \vdots \\ x_m \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \vec{b} \quad \text{where } B = [\vec{a}_1, \ldots, \vec{a}_m] \text{ invertible}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ \vdots \\ x_m \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ is basic feasible solution}$$

(ii) $p < m$:

Append $[\vec{a}_1, \ldots, \vec{a}_p]$ to $\underbrace{[\vec{a}_1, \vec{a}_2, \ldots \vec{a}_p, \ldots, \vec{a}_m]}_{\text{invertible}} = B$

Can be done $\because$ rank$(A) = m$.

then $$[\vec{a}_1, \ldots, \vec{a}_p ; \vec{a}_{p+1} \ldots \vec{a}_m \mid \underset{n-m}{R}] \begin{bmatrix} x_1 \\ \vdots \\ x_p \\ \hline 0 \\ \vdots \\ 0 \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix}_{n-m} = \vec{b}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ \vdots \\ x_p \\ 0 \\ \vdots \\ 0 \\ \hline 0 \\ \vdots \\ 0 \end{pmatrix} \text{ is a basic feasible solution (degenerate)}$$

(iii) $p > m$: impossible as rank$(A) = m$. Cannot have more than m linear independent columns

Case 2: $\{\vec{a}_1, \ldots, \vec{a}_p\}$ linear dependent

$\exists \alpha_i$, not all zero, $\alpha_1 \vec{a}_1 + \cdots + \alpha_p \vec{a}_p = \vec{0}$

If $\alpha_r \neq 0$, we can substitute $\vec{a}_r$ by other $\{\vec{a}_1, \ldots, \vec{a}_{r-1}, \vec{a}_{r+1}, \ldots \vec{a}_p\}$

i.e. $$\vec{a}_r = \frac{\alpha_1}{\alpha_r} \vec{a}_1 + \cdots + \frac{\alpha_{r-1}}{\alpha_r} \vec{a}_r + \frac{\alpha_{r+1}}{\alpha_r} \vec{a}_{r+1} + \cdots + \frac{\alpha_p}{\alpha_r} \vec{a}_p \quad ③$$

$x_r \cdot ③ \rightarrow ①$ $$\sum_{\substack{j=1 \\ j \neq r}}^{p} \left(x_j - x_r \frac{\alpha_j}{\alpha_r}\right) \vec{a}_j = \vec{b}.$$

Thus $$\vec{x} = \begin{pmatrix} x_1 - x_r \frac{\alpha_1}{\alpha_r} \\ \vdots \\ x_{r-1} - x_r \frac{\alpha_{r-1}}{\alpha_r} \\ 0 \\ x_{r+1} - x_r \frac{\alpha_{r+1}}{\alpha_r} \\ \vdots \\ x_p - x_r \frac{\alpha_p}{\alpha_r} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$ is a solution with one less nonzero entry

But $\vec{x}$ may no longer be feasible!

To make sure $\vec{x}$ is still feasible

$$\Leftrightarrow \quad x_j \geq x_r \frac{\alpha_j}{\alpha_r} \quad \forall j = 1, 2, \ldots, p. \quad \left(\begin{array}{c} j=r \text{ is} \\ \text{automatically} \\ \text{satisfied} \end{array}\right) \quad ④$$

Choice 1: Choose $\alpha_r > 0$ s.t.

$$\frac{x_r}{\alpha_r} = \min_{j=1,\ldots,p} \left\{ \frac{x_j}{\alpha_j} \;\middle|\; \alpha_j > 0 \right\} > 0 \quad \left(\because \begin{array}{c} x_j \geqq 0 \\ j=1,\ldots,p \end{array}\right)$$

$$\Leftrightarrow \quad \frac{x_j}{\alpha_j} \geq \frac{x_r}{\alpha_r} \text{ with } \alpha_j > 0 \Rightarrow ④ \text{ is true for } \alpha_j > 0$$

Clearly ④ is true for $\alpha_j < 0$, $(\because x_j, x_r, \alpha_r \geqq 0)$

Choice 2 choose $\alpha_r < 0$ s.t.

$$\frac{x_r}{\alpha_r} = \max_{j=1,\ldots,p} \left\{ \frac{x_j}{\alpha_j} \;\middle|\; \alpha_j < 0 \right\} < 0 \quad \left(\because \begin{array}{c} x_j \geqq 0 \\ j=1,\ldots,p \end{array}\right)$$

Thus by choosing eg. $\alpha_r > 0$ s.t.

$$\frac{x_r}{\alpha_r} = \min_{j=1,..,p} \left\{ \frac{x_j}{\alpha_j} \;\middle|\; \alpha_j > 0 \right\}$$

we get from

$$\begin{pmatrix} x_1 \\ \vdots \\ x_p \\ 0 \\ \vdots \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ 0 \\ \tilde{x}_{r+1} \\ \vdots \\ \tilde{x}_p \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow r^{\text{th}} \text{ entry} = 0.$$

FS of p nonzero entries → FS of $p-1$ nonzero entries

By repeating the argument, we can have

either (A) $\{\vec{a}_1, \vec{a}_2, \ldots \vec{a}_{r-1}, \vec{a}_r, \ldots, \vec{a}_p\}$ ... linear dependent

$\rightarrow$ repeat the argument in Case 2.

or (B) $\{\vec{a}_1, \vec{a}_2, \vec{a}_{r-1}, \vec{a}_r \ldots \vec{a}_p\}$ linear independent

$\rightarrow$ becomes Case 1

$\Rightarrow$ we can find a BFS.

Thm 2.3 BFS $\Rightarrow$ Extreme pt of FR.

pf: Let $\vec{x} = \begin{pmatrix} \vec{x}_B \\ 0 \end{pmatrix} \}m$ be BFS

i.e. (i) $[B \mid R] \begin{bmatrix} \vec{x}_B \\ \vec{0} \end{bmatrix} = \vec{b}$ with $B_{m \times m}$ invertible (Basic)

(ii) $\vec{x} \geq \vec{0}$ (Feasible)

Suppose by contradiction $\vec{x}$ not extreme pt of FR

$$\vec{x} = \lambda \vec{x}_1 + (1-\lambda) \vec{x}_2 \qquad \lambda \in (0,1)$$

where $\vec{x}_1 \neq \vec{x}_2 \in FR$, $\vec{x}_1 = \begin{pmatrix} \vec{u}_1 \\ \vec{v}_1 \end{pmatrix}$ $\vec{x}_2 = \begin{pmatrix} \vec{u}_2 \\ \vec{v}_2 \end{pmatrix}$

i.e. $\begin{cases} [B \mid R] \begin{bmatrix} \vec{u}_1 \\ \vec{v}_1 \end{bmatrix} = \vec{b} & [B \mid R] \begin{bmatrix} \vec{u}_2 \\ \vec{v}_2 \end{bmatrix} = \vec{b} \\ \vec{x}_1 \geq 0, & \vec{x}_2 \geq 0 \end{cases}$ ①

(A) $\because$ $\begin{pmatrix} \vec{x}_B \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} \vec{u}_1 \\ \vec{v}_1 \end{pmatrix} + (1-\lambda) \begin{pmatrix} \vec{u}_2 \\ \vec{v}_2 \end{pmatrix}$ $\lambda \in (0,1)$

with $\vec{v}_1 \geq 0$, $\vec{v}_2 \geq 0$

$\Rightarrow$ $\vec{v}_1 = \vec{v}_2 = 0.$ ②

(B) ①+②: $B \vec{u}_1 = B \vec{u}_2 = \vec{b}$

$\because$ B invertible $\vec{u}_1 = \vec{u}_2 = B^{-1} \vec{b}$

$\therefore$ $\vec{x}_1 = \vec{x}_2$ Contradiction

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Thm 2.4 Extreme pt of FR $\Rightarrow$ BFS

Pf: Let $\vec{x}_0 = \begin{pmatrix} x_1 \\ \vdots \\ x_r \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ be ex pt of FR

i.e. (i) $A\vec{x}_0 = \vec{b} \iff [\vec{a}_1, \dots, \vec{a}_r, \vec{a}_{r+1} \dots \vec{a}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_r \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{b}$

(ii) $x_i \geq 0 \quad i = 1, \dots, r$

Case 1: $\{\vec{a}_1, \dots, \vec{a}_r\}$ linear independent.

$\Rightarrow r \leq m = \text{rank}(A)$

$\Rightarrow \exists \{\vec{a}_{r+1}, \dots \vec{a}_m\}$ s.t.

$\{\vec{a}_1, \dots, \vec{a}_r, \vec{a}_{r+1} \dots \vec{a}_m\}$ l.i.

hence $[\underbrace{\vec{a}_1, \dots, \vec{a}_r, \dots \vec{a}_m}_{B \text{ invertible}} \mid R] \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_r \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{= \vec{x}_0} \begin{matrix} \\ \\ \\ \}m-r \\ \\ \end{matrix} = \vec{b}$

$\Rightarrow \vec{x}_0$ is a B.F.S.

Case 2 $\{\vec{a}_1, \dots \vec{a}_r\}$ linear dependent (Cannot happen proof by contradiction)

$\exists \alpha_r$ not all zero s.t. $\sum_{i=1}^{r} \alpha_i \vec{a}_i = \vec{0} \iff A\vec{\alpha} = \vec{0} \quad \vec{\alpha} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

define $\varepsilon = \min_{\substack{\alpha_i \neq 0 \\ 1 \leq i \leq r}} \frac{x_i}{|\alpha_i|} > 0$

$\left(\Rightarrow \frac{x_j}{|\alpha_j|} \geq \varepsilon \Rightarrow x_j \geq \varepsilon |\alpha_j| > 0 \quad \forall j = 1, \dots, r \right.$

$\left. \Rightarrow x_j \pm \varepsilon \alpha_j \geq 0 \right)$

Consider $\vec{x}_{\pm} = \vec{x}_0 \pm \varepsilon \vec{\alpha}$

(i) $(\vec{x}_{\pm})_j = (\vec{x}_0 \pm \varepsilon \vec{\alpha})_j = x_j \pm \varepsilon \alpha_j \geq 0 \quad \forall j = 1, \ldots, r$

$\therefore \vec{x}_{\pm} \geq 0$ feasible

(ii) $A \vec{x}_{\pm} = A(\vec{x}_0 \pm \varepsilon \vec{\alpha}) = A\vec{x}_0 \pm \varepsilon A\vec{\alpha} = \vec{b} + \vec{0} = \vec{b}$

$\therefore \vec{x}_{\pm}$ is solution

hence $\vec{x}_{\pm} \in FR$

(iii) $\vec{x}_0 = \frac{1}{2}\vec{x}_+ + \frac{1}{2}\vec{x}_-$

$\Rightarrow \vec{x}_0$ is not ex. pt of FR contradiction.

Thus Case 2 will never happen.

Thm 2.5 Optimal solution occurs at ex pt. of FR.

Pf: Let $\vec{x}_0 \in FR$ be the optimal solution

(i) $A\vec{x}_0 = \vec{b}$

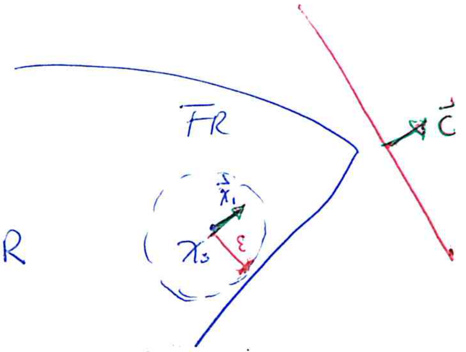
(ii) $\vec{x}_0 \geq \vec{0}$

(iii) $z_0 = \vec{c}^T\vec{x}_0 \geq \vec{c}^T\vec{x} \quad \forall \vec{x} \in FR$

Case 1: $\vec{x}_0$ cannot be in the interior of FR (proof by Contradiction)

If $\vec{x}_0$ is in interior of FR

$\exists\, B_\varepsilon(\vec{x}_0) \subseteq FR$



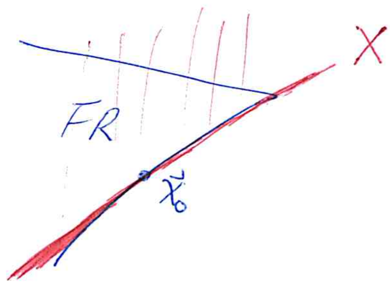
Consider $\vec{x}_1 = \vec{x}_0 + \frac{\varepsilon}{2}\frac{c}{|c|} \in B_\varepsilon(\vec{x}_0) \subseteq FR$

$\vec{c}^T\vec{x}_1 = \vec{c}^T\vec{x}_0 + \frac{\varepsilon}{2}\frac{\vec{c}^T c}{|c|}$

$= z_0 + \frac{\varepsilon}{2}|c| \gneq z_0$ (Contradiction to optimality of $\vec{x}_0$, i.e. (iii))

Case 2 $\vec{x}_0$ is boundary pt of FR

Let $X \equiv \{\vec{x} \mid \vec{c}^T\vec{x} = z_0\}$



(iv) $\vec{x}_0 \in X$

(v) $FR \in X^- \equiv \{\vec{x} \mid \vec{c}^T\vec{x} \leq z_0\}$ ($\because$ (iii))

$\therefore$ X is a supporting hyperplane of FR at $\vec{x}_0$

Since FR is bounded from below ($\because \vec{x} \geq 0$)

$\therefore \exists$ ex. pt of FR (Thm 1.5)